

Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R^4

M. Hayakawa *

Theory Division, KEK, Tsukuba, Ibaraki 305-0801, Japan

December 21, 1999

Abstract

Here we discuss the ultraviolet and infrared aspects of the noncommutative counterpart of QED, which is called as noncommutative QED, as well as some infrared dynamics of noncommutative Yang-Mills (NCYM) theory. First we demonstrate that the divergence in the theory can be subtracted by the similar counterterms as in ordinary theory at one loop level. Then the anomalous magnetic moment is calculated to see the infrared aspect of the theory which reflects the violation of Lorentz symmetry. The evaluation of the finite part of the photon vacuum polarization shows that the logarithmically singular term in the infrared limit appears with the same weight as UV logarithmic divergence, showing the correlation between the UV and infrared dynamics in NCYM theory. NC-QED theory does not show such a property. We also consider the extension to chiral gauge theory in the present context, but the requirement of anomaly cancellation allows only noncommutative QED.

1 Introduction

The analysis of quantum mechanical features of noncommutative field theory is now being developed from purely field theoretical point of view [1–12]. The common character of noncommutative field theories is its nonlocality. The product of the operators on the noncommutative geometry is mapped into the starred (referred to hereafter as $*$ -) product on the deformed algebra of the usual functions. This gives the above momentum-dependent phase factor for each interaction vertex which manifests nonlocality of the theory. This phase factor selects the diagrams with the ultraviolet (UV) divergence since it can serve as an effective cutoff of high frequency modes [2, 6, 9, 10, 12]. As a consequence the UV-divergent diagrams correspond to the 'planar' diagrams; no phase suppression factors in the noncommutative theory side; the 't Hooft diagrams [13] which can be drawn on a plane in the ordinary large N SU(N) gauge theory side (a product of plaquette variables with no Z_N -phase factors in the twisted reduced model [1]). Thus noncommutative Yang-Mills (NCYM) theory would be equivalent to the large N ordinary

*electric address: haya@post.kek.jp

Yang-Mills system at least in the high momentum region as argued at the level of Feynman diagram [6, 9].

The paper intends to describe the ultraviolet aspects of noncommutative QED (NC-QED) in Sec. 3 which has not been reported in short article [12] as well as the detail about the computation of the infrared aspects common to NCYM theory in Sec. 4 and Appendix. To make this paper self-contained and explain the results more clearly, the other parts of Ref. [12] should also be described in detail. It is expected to guide and help one to further study the quantum mechanical aspect of noncommutative field theory.

Thus the paper is organized as follows: Sec. 2 is concerned with incorporation of the matter fields and showing that the allowed choice is quite limited. In Sec. 3 NC-QED theory is quantized to investigate ultraviolet aspect to demonstrate that the one-loop divergence can be subtracted by the usual set of local counterterms. Sec. 4 studies the infrared aspects of the theory through the anomalous magnetic dipole moment and vacuum polarization of photon. The extension to the chiral gauge theory is also examined in Sec. 5, but it is found that there is *no* chiral gauge theory. Sec. 6 is devoted to the summary of the present paper.

2 Construction of Classical Action

We begin with pure U(1) NCYM system, which has the classical action in space-time dimension d

$$S_{YM} = \int d^d x \left(-\frac{1}{4g^2} \right) F_{\mu\nu} * F^{\mu\nu}. \quad (1)$$

Here the field strength $F_{\mu\nu}$ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_M, \quad (2)$$

where $[A, B]_M$ denotes Moyal bracket:

$$[A, B]_M = A * B - B * A. \quad (3)$$

The $*$ -product

$$A * B(x) \equiv e^{\frac{1}{2i} C^{\mu\nu} \partial_\mu^{(\xi)} \partial_\nu^{(\eta)}} A(x + \xi) B(x + \eta) \Big|_{\xi, \eta \rightarrow 0}, \quad (4)$$

is defined in terms of the infinite towers of derivative operation with use of an antisymmetric matrix $C^{\mu\nu}$ ($C^{\mu\nu}$ has the dimension of area.) which reflects noncommutativity of space-time by modifying the algebra of functions. Even in U(1) case A_μ couples to itself since the field strength $F_{\mu\nu}$ has the nonlinear term in A_μ . The $*$ -product obeys the associative law which is also satisfied by the matrix algebra, Thus the algebraic manipulation in $*$ -product is the same as in the calculus of matrix. Then, assuming that the fields decrease so promptly at infinity that the space-time integral of a Moyal bracket (which corresponds to the trace of the commutator in the “matrix language”) vanishes, it is easily shown that the action (1) is invariant under the gauge transformation

$$A_\mu(x) \rightarrow A'_\mu(x) = U(x) * A_\mu(x) * U^{-1}(x) + iU(x) * \partial_\mu U^{-1}(x), \quad (5)$$

where $U(x) = (e^{i\theta(x)})_*$ is defined by an infinite series of multiple $*$ -product of scalar function $\theta(x)$. The similarly defined $U^{-1}(x) = (e^{-i\theta(x)})_*$ corresponds to the inverse of $U(x)$. The coupling of “electron” to gauge field in U(1) NCYM theory receives a severe restriction. Here we mean by ”electron”, the fields which couple to the gauge fields in the similar manner as in ordinary QED, in which the covariant derivative for the matter field ψ with charge Q_ψ is given by

$$D_\mu\psi = \partial_\mu\psi - iQ_\psi A_\mu\psi. \quad (6)$$

Conventional assignment of charge Q_ψ in the electroweak theory is like $Q_e = -1$ for electron, $Q_u = \frac{2}{3}$ for up-type quark, etc. As is well-known there is no reason for Q_ψ to be quantized in the electroweak theory. Extension to noncommutative case shows that nonzero U(1) charge is not only quantized, but also limited to be $+1$ or -1 even on Minkowski space-time. This contrasts with the mechanism of charge quantization on compact manifold from the single-valuedness of the wave function. We show those facets below.

First a simple algebraic manipulation shows that a combination

$$D_\mu\psi = \partial_\mu\psi - iA_\mu * \psi, \quad (7)$$

which resembles (6), behaves covariantly

$$D_\mu\psi(x) \rightarrow D'_\mu\psi'(x) = U(x) * D_\mu\psi(x), \quad (8)$$

under transformation (5) for $A_\mu(x)$ and

$$\psi(x) \rightarrow \psi'(x) = U(x) * \psi(x), \quad (9)$$

for $\psi(x)$. Likewise if $\hat{\psi}(x)$ transforms as

$$\hat{\psi}(x) \rightarrow \hat{\psi}'(x) = \hat{\psi}(x) * U^{-1}(x), \quad (10)$$

then the quantity

$$D_\mu\hat{\psi} = \partial_\mu\hat{\psi} + i\hat{\psi} * A_\mu, \quad (11)$$

behaves in the same manner as (10). Commutative limit $C^{\mu\nu} \rightarrow 0$ indicates that $\hat{\psi}$ corresponds to a field in ordinary U(1) gauge theory with the same magnitude but opposite sign compared to the one associated with ψ . Thus we say that $\hat{\psi}$ has charge -1 while ψ carries $+1$.

Therefore, for instance, the action

$$S_{\text{matter}} = \int d^d x (\bar{\psi} * \gamma^\mu i D_\mu \psi - m \bar{\psi} * \psi), \quad (12)$$

has local U(1) invariance since $\bar{\psi}$ transforms as $\hat{\bar{\psi}}$. However a simple exercise shows that the simple extension

$$D_\mu\psi^{(n)} = \partial_\mu\psi^{(n)} - inA_\mu * \psi^{(n)}, \quad (13)$$

with $\psi^{(n)} \rightarrow \psi^{(n)'} = U^n \psi^{(n)}$ for the field ψ with integral multiple n of unit charge fails to transform covariantly.

Derivation of the above facts and the various formula is similar to that in ordinary non-abelian gauge theory due to the simple fact that $*$ -product satisfies the associative law. The field with charge $+1(-1)$ in noncommutative case would correspond to (anti-)fundamental representation in ordinary nonabelian gauge theory. It is also reminiscent of such features that noncommutative gauge theory carries the internal degrees of freedom by imbedding them into the space-time geometry itself. This is the reverse process of the reduction of the space-time degrees of freedom into the internal ones in the large N gauge theory [14, 1]. When we pursue this correspondence further, we are inclined to guess that the higher-rank representation of $SU(N)$ gauge theory may convert into some matter fields in noncommutative gauge theory. It would be the counterpart of the fields with an integral multiple of unit charge from the view point of noncommutative generalization of $U(1)$ gauge theory. Actually the adjoint representation corresponds to a field $\chi(x)$ with zero charge in total but transforming in the by-product form

$$\chi(x) \rightarrow \chi'(x) = U(x) * \chi(x) * U^{-1}(x). \quad (14)$$

Its covariant derivative is given by Moyal bracket (3). What is the counterpart, for instance, of the second-rank antisymmetric representation of $SU(N)$ gauge theory? We could not succeed to find those other counterparts. Although a consequence depends on groping the possibility, we see that the fields of nonzero charge with different absolute magnitude cannot coexist in a system.

Note that the above argument also persists for the static electron. Thus the the vacuum expectation value of Wilson loop along the simple rectangular loop is associated with the ground energy acting between the static sources with charges ± 1 similarly as in ordinary Yang-Mills theory.

3 Perturbation Theory of Noncommutative QED

To perform perturbation theory for NC-QED, we rescale the gauge field appearing in the previous section as $A_\mu \rightarrow gA_\mu$. To perform gauge fixing to obtain the nonsingular free propagator for the gauge fields, BRST quantization similar to Ref. [3] is adequate. Incorporating the ghost fields c, \bar{c} and the auxiliary fields B , BRST transformation

$$\begin{aligned} \hat{\delta}_{\mathbf{B}} A_\mu &= D_\mu c = \partial_\mu c - ig[A_\mu, c]_{\mathbf{M}}, \\ \hat{\delta}_{\mathbf{B}} c &= igc * c, \quad \hat{\delta}_{\mathbf{B}} \bar{c} = iB, \quad \hat{\delta}_{\mathbf{B}} B = 0 \\ \hat{\delta}_{\mathbf{B}} \psi &= ic * \psi, \quad \hat{\delta}_{\mathbf{B}} \bar{\psi} = i\bar{\psi} * c \end{aligned} \quad (15)$$

is nilpotent. Under this transformation the quantities in (1) and (12) are BRST-closed respectively. The Faddeev-Popov and gauge fixing term is introduced as BRST-exact form

$$S_{\text{GF}} = \int d^d x (-i\hat{\delta}_{\mathbf{B}}) \frac{1}{2} \left(\bar{c} * \left(\frac{\alpha}{2} B + \partial_\mu A^\mu \right) + \left(\frac{\alpha}{2} B + \partial_\mu A^\mu \right) * \bar{c} \right). \quad (16)$$

After elimination of B through its equation of motion the above quantity reduces to

$$S_{\text{GF}} = \int d^d x \left(-\frac{1}{2\alpha} \partial_\mu A^\mu * \partial_\nu A^\nu + \frac{1}{2} (i\bar{c} * \partial^\mu D_\mu c - i\partial^\mu D_\mu c * \bar{c}) \right). \quad (17)$$

We consider NC-QED on Minkowski space-time to estimate the on-shell electron coupling to the photon. To simplify the discussion \ast -product works effectively on one plane in the space in the canonical basis of $C^{\mu\nu}$. This is, we consider the case when $C^{01} = 0$ but nonzero C^{23} . Then there are two kinds of on shell photon; one with momentum along (2,3)-plane; one with that moving orthogonal to (2,3)-plane. Thus we can discuss the effect of explicit Lorentz invariance introduced by C^{23} .

As in Ref. [3] quantization of noncommutative theory is realized as a perturbative expansion by giving Feynman rules through

$$Z[J] = \int D\Phi e^{iS[\Phi] + i \int d^d x J(x) \ast \Phi(x)}, \quad (18)$$

where the action S is the sum of (1), (12)

$$S_{\text{NC-QED}} = \int d^d x \left(-\frac{1}{4g^2} F_{\mu\nu} \ast F^{\mu\nu} + \bar{\psi} \ast \gamma^\mu i D_\mu \psi - m \bar{\psi} \ast \psi \right). \quad (19)$$

together with (17). As a result the propagator is the same as in its commutative counterpart, while each vertex accompanies with a phase factor shown in Fig. 1 depending on the momenta outgoing from the vertex. It manifests the nonlocal nature of theory.

In order to pursue ultraviolet (UV) divergent structure, we begin with discussing a few typical vertex functions (contribution to Green functions from one particle irreducible Feynman diagrams with all the external legs amputated) in the succeeding subsections. This would be the prompt approach to learn about the nature of our theory. The aim of our analysis below is to demonstrate that at one-loop level the present theory can be made finite by redefining the fields and parameters consistently with the symmetry of the classical action. The analysis also shows that UV divergence can appear only in a restricted set of diagrams, “planar diagram” which will be defined at the end of Sec. 3.2.

3.1 Two point functions

The electron self-energy receives one-loop correction through only one diagram shown in Fig. 2 as in ordinary QED. In this diagram the phase factors accompanied with the two vertices cancel with each other. Thus the contribution is nothing but the one found in ordinary QED. Thus not only UV divergence can be subtracted by the usual rescaling of wave function and mass of electron, but also the remained finite part does not change as long as the same renormalization scheme ($\overline{\text{MS}}$ -scheme here) is applied. The wave function renormalization factor Z_ψ for the electron is in particular found as

$$Z_\psi = 1 - \frac{g^2}{16\pi^2} \frac{1}{\varepsilon'}, \quad (20)$$

where $1/\varepsilon' = 1/\varepsilon + \gamma_E - \ln(4\pi)$ for the space-time dimension $d = 4 - 2\varepsilon$. Here and hereafter Feynman gauge ($\alpha = 1$) is taken to simplify the expression.

The photon self-energy diagram receives an electron loop contribution (Fig. 3(d)) as well as the

$$\begin{aligned}
& \text{Diagram 1: } \mu \text{ wavy line, } p_I, p_F \text{ fermion lines} \quad = i g \gamma^\mu \exp\left(\frac{i}{2} p_I C p_F\right) \\
& \text{Diagram 2: } \mu_1, \mu_2, \mu_3, \mu_4 \text{ wavy lines, } p_1, p_2, p_3, p_4 \text{ momenta} \quad = -2g \sin\left(\frac{1}{2} p_1 C p_2\right) \\
& \quad \times [(p_1 - p_2)^{\mu_3} g^{\mu_1 \mu_2} + (p_2 - p_3)^{\mu_1} g^{\mu_2 \mu_3} + (p_3 - p_1)^{\mu_2} g^{\mu_3 \mu_1}] \\
& \text{Diagram 3: } \mu_1, \mu_2, \mu_3, \mu_4 \text{ wavy lines, } p_1, p_2, p_3, p_4 \text{ momenta} \quad = -4ig^2 [(g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}) \\
& \quad \times \sin\left(\frac{1}{2} p_1 C p_2\right) \sin\left(\frac{1}{2} p_3 C p_4\right) + (g^{\mu_1 \mu_4} g^{\mu_2 \mu_3} - g^{\mu_1 \mu_2} g^{\mu_3 \mu_4}) \\
& \quad \times \sin\left(\frac{1}{2} p_3 C p_1\right) \sin\left(\frac{1}{2} p_2 C p_4\right) + (g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_3} g^{\mu_2 \mu_4}) \\
& \quad \times \sin\left(\frac{1}{2} p_1 C p_4\right) \sin\left(\frac{1}{2} p_2 C p_3\right)] \\
& \text{Diagram 4: } \mu \text{ dashed line, } p_I, p_F \text{ fermion lines} \quad = 2ig p_F^\mu \sin\left(\frac{1}{2} p_I C p_F\right)
\end{aligned}$$

Figure 1: Feynman rule in noncommutative QED

FP ghost (Fig. 3(a)) and gauge boson (Fig. 3(b), (c)) loops which already appear in NCYM theory. The evaluation of the diagram is illustrated for the ghost loop diagram in Fig. 3(a) in Appendix. Here at first we take up the simpler diagram, the gauge boson diagram in Fig. 3(c)

$$i\Pi_{(c)}^{\mu\nu}(q) = 2(d-1)ig^2 g^{\mu\nu} \int_0^\infty id\alpha \frac{1}{[4\pi\alpha i]^{d/2}} \left(1 - \exp\left[-i\frac{1}{\alpha}\frac{\tilde{q}^2}{4}\right]\right). \quad (21)$$

In dimensional regularization the part which does not depend on any phase factor vanishes (no divergence). On the other hand, the portion of (21) arising from the contribution dependent on a loop momentum (nonplanar contribution) evades the integral to diverge due to an effective damping factor, $\exp\left(-i\frac{1}{\alpha}\frac{\tilde{q}^2}{4}\right)$. Thus it is finite

$$i\Pi_{(c)}^{\mu\nu}(q) = i\frac{g^2}{16\pi^2} g^{\mu\nu} \frac{-24}{-\tilde{q}^2}, \quad (22)$$

which shows the $1/\tilde{q}^2$ singularity for $\tilde{q}^2 \rightarrow 0$. It should be remembered that such a portion of (21) that leads to (22) became finite due to finiteness of \tilde{q}^2 . But it would lead ultraviolet

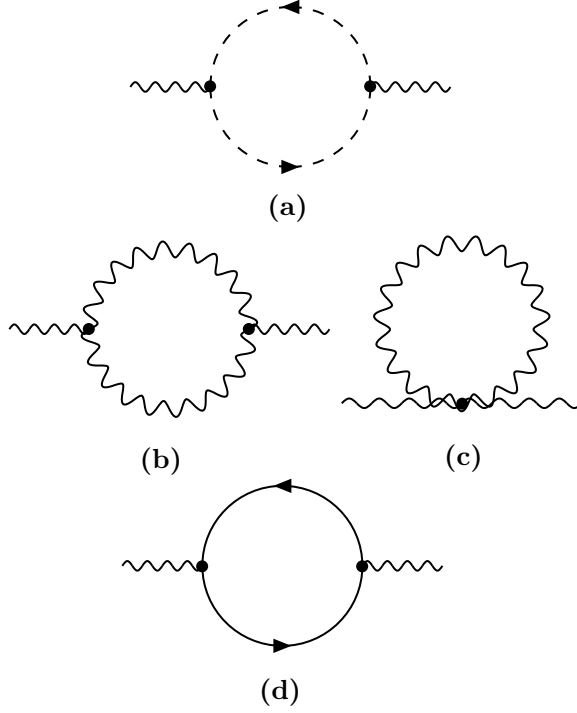


Figure 3: Correction to vacuum polarization of photon

two topologically distinct diagrams which contribute to this interaction. The first one shown in Fig. 4(a) is the analogue of QED while the other occurs through a nonabelian vertex involving three gauge bosons as shown in Fig. 4(b). The QED-like diagram in Fig. 4(a) is finite while the nonabelian-type diagram in Fig. 4(b) leads UV-divergence. The local counterterm with the coefficient

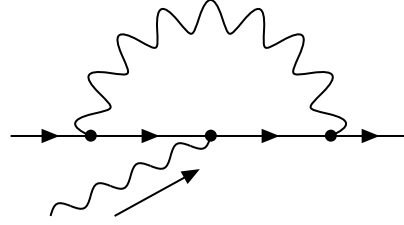
$$Z_{\bar{\psi}\psi A} = 1 - \frac{g^2}{16\pi^2} \frac{1}{\epsilon'} \times 3, \quad (26)$$

is sufficient to remove this divergence.

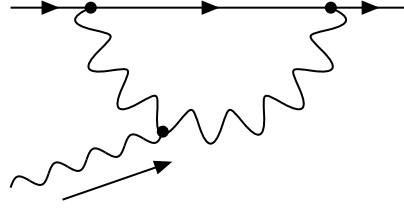
Here we would like to check whether we identify UV divergence properly, and quantize a system maintaining local gauge invariance. The nontrivial point to be examined here is the universality of gauge coupling constant. This implies the relations among the rescale factors Z_ϕ for each field ϕ , $Z_{\phi_1\phi_2\phi_3}$ for each coupling to which ϕ_1 , ϕ_2 and ϕ_3 participate

$$\frac{Z_{\bar{\psi}\psi A}}{Z_\psi} = \frac{Z_{AAA}}{Z_A} = \sqrt{\frac{Z_{AAAA}}{Z_A}} = \frac{Z_{\bar{c}cA}}{Z_c}. \quad (27)$$

Note that $Z_{\bar{c}cA}$ and Z_c do not alter even after incorporation of electrons at one-loop level. Thus



(a) One loop QED-like correction to $\psi\bar{\psi}A_\mu$



(b) Nonabelian-type correction to $\psi\bar{\psi}A_\mu$

Figure 4: Correction to $\psi\bar{\psi}A_\mu$ vertex

the final combination found in eq. (27) is completely the same as in NCYM case

$$\frac{Z_{\bar{c}cA}}{Z_c} = 1 - \frac{g^2}{16\pi^2} \frac{1}{\varepsilon'} \times 2. \quad (28)$$

The relation (27) and this fact insist that the effects of incorporation of electron must cancel between the denominator and the numerator in each of the other three combinations in (27). Before observing this fact we recall the situation in ordinary QED. In ordinary QED the relation (27) reduces to

$$\left. \frac{Z_{\bar{\psi}\psi A}}{Z_\psi} \right|_{\text{QED}} = 1. \quad (29)$$

That is, $Z_\psi|_{\text{QED}}$ is required to balance completely with $Z_{\bar{\psi}\psi A}|_{\text{QED}}$. $Z_{\bar{\psi}\psi A}|_{\text{QED}}$ is derived from UV-divergence of the similar diagram in Fig. 4(a). Returning to NC-QED, the argument in Sec. 3.1 showed that Z_ψ is the same (20) found in ordinary QED, $Z_\psi = Z_\psi|_{\text{QED}}$. However in NC-QED the QED-like diagram in Fig. 4(a) does not diverge contrary to the case in QED. (Note that this feature persists for any choice of gauge fixing parameter.) Those facts raise the question whether the relation (27) is actually maintained or not. Somewhat curiously $Z_{\bar{\psi}\psi A}$ in eq. (26) obtained from Fig. 4(b) and Z_ψ in (20) establishes eq. (27) with (28). The direct computation also shows that all the divergences in AAA- and AAAA-vertices can be subtracted

by the usual counterterms and the associated Z factors satisfy the relations required in eq. (27). Also in the present theory, the β function signifies whether the theory is well defined at an UV fixed point or not. It also helps one to improve the accuracy of the approximation by resumming the large logarithmic corrections. However we must always keep in mind that the typical behavior at some fixed energy scale cannot be drawn from the usual intuition based on the block-spin renormalization group method applicable to local field theory.

The beta function is the simplest to find by observing the gauge interaction of Faddeev-Popov ghosts. There only Z_A derived from the singularity of the photon vacuum polarization receives the electron effect at one-loop level. Eqs. (28) and (25) lead

$$\beta(g) = \frac{1}{g} Q \frac{dg}{dQ} = - \left(\frac{22}{3} - \frac{4}{3} N_F \right) \frac{g^2}{16\pi^2}. \quad (30)$$

A contribution $\frac{22}{3}$ is due to the structure similar to nonabelian dynamics which is pure SU(2) Yang-Mills theory [3, 4]. However the matter contribution is that found in QED theory with unit charge, *not* that of the quarks belonging to the fundamental representation of SU(2) gauge theory (where $\frac{2}{3}$ instead of $\frac{4}{3}$ per flavor in (30)).

The above analysis indicates the following general features about UV divergence. First we write a diagram according to Feynman rule in Fig. 1. Then the associated contributions are divided into the pieces each of which has a definite momentum-dependent phase factor. If a given piece admits a loop momentum which does not appear in any phase factor, it can lead divergence (It may be a subdivergence which should be subtracted by the counterterms determined at the previous order of perturbation.) We define such a diagram as “planar” diagram. Power counting argument allows us to choose the superficially divergent pieces among those planar diagrams. Since the same mechanism above works in the more complex diagrams appearing in the vertex functions with three or more external legs, UV divergence can only appear in the vertex functions corresponding to that in the ordinary QCD (three gauge boson vertices, etc.). Determination of the above fact due to ’t Hooft-type representation may be developed along the same line as done for NCYM theory in Ref. [2, 9, 10].

4 Some Infrared Aspects

We are more interested in the infrared aspect under the situation that UV behavior is shown to be greatly modified due to nonabelian nature of noncommutative field theory. The small number of flavors insures that the theory is asymptotically free. This behavior suggests the strong coupling low energy dynamics of the theory. But here we assume that the coupling is kept small and the theory admits the perturbative analysis. The perturbative infrared aspect is examined by calculating the leading correction to magnetic dipole moment ($g - 2$) in Sec. 4.1 as well as the vacuum polarization of the photon in Sec. 4.2, and ask what modification occurs in the extension to NC-QED.

4.1 Anomalous magnetic dipole moment

The extraction of dipole coupling from Figs. 4(a) and 4(b) yields

$$ig^3 \left[e^{\frac{i}{2}p_I \cdot C \cdot p_F} H(1, p, q) + e^{\frac{i}{2}p_I \cdot C \cdot p_F} H(0, p, q) - e^{-\frac{i}{2}p_I \cdot C \cdot p_F} H(1, p, q) \right] m i \sigma^{\mu\nu} q_\nu, \quad (31)$$

where q is the incoming photon momentum, and p is connected to the incoming electron momentum p_I and the outgoing electron momentum p_F through

$$p_I = p - \frac{q}{2}, \quad p_F = p + \frac{q}{2}. \quad (32)$$

The matrix $\sigma^{\mu\nu}$ is here $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. The contribution in the first line in eq. (31) is from Fig. 4(a) and one in the second line from Fig. 4(b). The function $H(\eta, p, q)$ appearing in (31) is

$$\begin{aligned} H(\eta, p, q) = & \int_0^\infty id\alpha_0 \int_0^\infty id\alpha_+ \int_0^\infty id\alpha_- \frac{1}{[4\pi\beta i]^2} \\ & \times 2 \left(\frac{\alpha_+ + \alpha_-}{\beta} - \left(\frac{\alpha_+ + \alpha_-}{\beta} \right)^2 \right) \\ & \times \exp \left[-i \frac{1}{\beta} \left\{ (\alpha_+ + \alpha_-)^2 m^2 + \alpha_+ \alpha_- (-q^2) \right. \right. \\ & \left. \left. - \eta (\alpha_+ + \alpha_-) (p \cdot \tilde{q}) + \eta^2 \frac{\tilde{q}^2}{4} \right\} \right], \end{aligned} \quad (33)$$

where $\beta = \alpha_0 + \alpha_+ + \alpha_-$ and $\tilde{q}^\mu = C^{\mu\nu} q_\nu$ has the dimension of length. $H(0, p, q)$ is $\frac{1}{16\pi^2 m^2}$ for on-shell photon. If the naive commutative limit ($C^{\mu\nu} \rightarrow 0$) were taken before integration two contributions (the second line in eq. (31)) from Fig. 4(b) would cancel with each other, leaving the usual contribution to $(g - 2)$ from the first graph, Fig. 4(a). The calculation similar to Appendix leads

$$\begin{aligned} H(1, p, q) = & \frac{1}{8\pi^2} \int_0^1 d\alpha_+ \int_0^{(1-\alpha_+)} d\alpha_- \frac{(\alpha_+ + \alpha_-) - (\alpha_+ + \alpha_-)^2}{(\alpha_+ + \alpha_-)^2 m^2 + \alpha_+ \alpha_- (-q^2)} \\ & \times e^{i(\alpha_+ + \alpha_-)(p \cdot \tilde{q})} x K_1(x), \end{aligned} \quad (34)$$

where $x = (-\tilde{q}^2) \{ (\alpha_+ + \alpha_-)^2 m^2 + \alpha_+ \alpha_- (-q^2) \}$, and $K_1(x)$ is a modified Bessel function of the second kind [15]. At this stage we can justify that q^2 and \tilde{q}^2 in $H(1, p, q)$ is brought to zero without confronting with any singularities since $K_1(x) \sim \frac{1}{x}$ for $x \sim 0$. Thus for $q^2 = 0$ and $\tilde{q}^2 = 0$, $H(1, p, q)$ becomes equal to $H(0, p, q)$. Therefore the magnetic dipole moment does not change for the photon with no transverse momentum along (2, 3)-plane ($\tilde{q} = 0$). The photon on mass shell can, of course, have the momentum transverse along such a plane. Eq. (31) then exhibits the difference where

$$H(1, p, q)|_{q^2=0} = \frac{1}{8\pi^2 m} \int_0^1 ds (1-s) e^{is(p \cdot \tilde{q})} x(s) K_1(x(s)), \quad (35)$$

with $x(s) = (-\tilde{q}^2)m^2s^2$. We note the two points. The first is that the third term in eq. (31) carries the phase which should be accompanied by the fields with a charge opposite to the electron. The origin of the sign change can be easily understood in terms of the double-line representation of photon propagation [2, 9, 10]. The second feature is that nonzero \tilde{q}^2 enters in the integrand with a combination $x = (-\tilde{q}^2) \{(\alpha_+ + \alpha_-)^2 m^2 + \alpha_+ \alpha_- (-q^2)\}$, or $(-\tilde{q}^2)(-q^2)$. Such a feature is also observed in the charge form factor. This indicates that the threshold behavior around electron-positron pair production $q^2 \sim 4m^2$ is largely affected by nonzero \tilde{q}^2 .

4.2 Vacuum polarization of photon

The calculation for the finite part of the vertex functions will be necessary to compute the cross section of, for instance, e^+e^- annihilation process in NC-QED. One important vertex function is the photon vacuum polarization. Indeed the following one-loop calculation of it gives us the interesting information on the quantum mechanical dynamics of noncommutative field theory. Here we first concentrate on the contributions from Fig. 3(a) \sim (c), which also exists in NCYM theory.

We first remind the appearance of the singular term $\sim 1/\tilde{q}^2$ found in eq. (22) for one Feynman diagram in Fig. 3(c). The question is whether it remains even after summing up Fig. 3(a) \sim (c). The formula (52) derived for the integrals in Appendix enables us to extract the singular terms $1/\tilde{q}^2$ and $\ln(\tilde{q}^2 q^2)$ from eq. (23)

$$i\Pi_{(a)+(b)}^{\mu\nu} \sim i \frac{g^2}{16\pi^2} \left\{ g^{\mu\nu} \frac{24}{-\tilde{q}^2} + \frac{10}{3} (g^{\mu\nu} q^2 - q^\mu q^\nu) \ln(q^2 \tilde{q}^2) + 32 \frac{\tilde{q}^\mu \tilde{q}^\nu}{\tilde{q}^4} - \frac{4}{3} \frac{q^2}{\tilde{q}^2} \tilde{q}^\mu \tilde{q}^\nu \right\}. \quad (36)$$

In the sum of (22) and (36) $g^{\mu\nu}/\tilde{q}^2$ term cancels out

$$i\Pi^{\mu\nu}(q) \sim i \frac{g^2}{16\pi^2} \left\{ \frac{10}{3} (g^{\mu\nu} q^2 - q^\mu q^\nu) \ln(q^2 \tilde{q}^2) + 32 \frac{\tilde{q}^\mu \tilde{q}^\nu}{\tilde{q}^4} - \frac{4}{3} \frac{q^2}{\tilde{q}^2} \tilde{q}^\mu \tilde{q}^\nu \right\}, \quad (37)$$

which is consistent with Slavnov-Taylor identity derived from BRST symmetry. It should be recalled that the nonplanar contribution would diverge if the integral in eq. (23) were evaluated with \tilde{q}^2 set equal to zero. The logarithmic infrared singularity $\ln(\tilde{q}^2)$ in (37) reflects the fact that UV divergence is at most logarithmic. In fact the coefficient $10/3$ of $\ln(\tilde{q}^2)$ is that of the wave function renormalization factor (24) of photon in NCYM theory. Thus the term singular in \tilde{q}^2 correlates with the ultraviolet divergence. Such a phenomenon is also observed in Ref. [6, 10]. The other terms proportional to $\tilde{q}^\mu \tilde{q}^\nu$ is interesting. The effect of such terms is obscure until the concrete cross section is estimated.

NC-QED has one extra contribution from Fig. 3 (d). It is entirely planar as was discussed in Sec. 4.2. Thus the logarithmic infrared singularity in the one-loop correction to photon vacuum polarization of NC-QED theory is completely the same as in NCYM theory at one-loop level. On the other hand the wave function renormalization of photon receives a net effect from such a planar contribution and results in eq. (25). Thus NC-QED theory does not have the infrared-UV correspondence as found in NCYM theory.

5 Weyl Fermions and Chiral Gauge Theory

Until now all the fermions are assumed to be Dirac fermions. It is naturally tempted to pursue the extension to chiral gauge theory. Since the classical analysis given in Sec. 2 is irrelevant to the chiral property of fermion, Weyl fermions can have the charge +1 or -1. The right-handed fermion with +1 charge is easily seen to be replaced by its CP conjugate (the left-handed) fermion also in the present context. Thus the chiral gauge theory simply implies that the number of the left-handed fermions with +1 charge is not equal to that with -1¹. The question is whether such a theory circumvents a triangular loop anomaly to define a consistent quantum theory or not.

Let us imagine one triangular diagram in which the fermion number flows from 1 to 2 with three incoming external momentum q_i ($i = 1, 2, 3$). Then the momentum conservation gives q_i in terms of the internal momenta

$$q_1 = k_2 - k_3, \quad q_2 = k_1 - k_2, \quad q_3 = k_2 - k_3. \quad (38)$$

The phase factor associated with the diagram in our mind is

$$\exp \left[\frac{i}{2} (k_3 \cdot C \cdot k_2 + k_2 \cdot C \cdot k_1 + k_1 \cdot C \cdot k_3) \right]. \quad (39)$$

With help of eq. (38) this phase factor reduces to $e^{-\frac{i}{2}q_1 \cdot C \cdot q_2}$. Thus a triangular fermion loop diagram gives only planar contributions. Once we remind the correspondence between the current theory to ordinary nonabelian gauge system in which the external momentum plays the role of color in the gauge theory side, the remained integral is evaluated in the same manner as in ordinary nonabelian gauge theory which involves the fundamental and/or anti-fundamental Weyl fermions. From this observation, the number of the left-handed fermions with -1 charge match with the number of +1 in the system. Such a theory is vector-like, i.e., nothing but NC-QED considered until the previous sections.

6 Summary

In this paper NC-QED, which is the simplest extension of U(1) NCYM theory, is studied to observe the UV structure and the perturbative aspects of the low momentum region. It is observed that one loop diagram are all made finite by the redefinition of the operators and parameters originally involved in a classical action. From the evaluation of those one loop diagrams, the UV divergence is suggested to appear only in the planar diagram, which is the same feature shared already in NCYM theory. Although NCYM theory accommodates SU(2)-like structure in UV divergence, U(1) facet appears in β function when the electron is introduced. The possibility of the extension to chiral gauge theory is examined, but the simple extension

¹We require that the triangular loop contribution cancels with each other for *all* momentum configuration. But it might be too strong requirement for noncommutative SU(N) gauge theory due to non-factorizability of color and phase factors, as suggested by Y. Kitazawa.

does not admit any chiral gauge theory.

To observe the infrared behavior, the anomalous magnetic dipole moment is evaluated in this theory. The value is the same as in conventional QED for the photon propagating in the direction orthogonal to the plane on which noncommutativity enters. In the transverse direction, the magnetic coupling depends on the magnitude of the components of the momentum along this direction. This exhibits the violation of full Lorentz invariance $SO(1,3)$ through the breaking parameter C^{23} . It is desirable to know the next-order correction which will pick up the one-loop modification to the photon propagator. Also the explicit evaluation of the cross section of e^+e^- annihilation process, Compton scattering process, etc., should be carried out to capture the relationship with this explicit violation more directly.

The finite part of the photon vacuum polarization indicates that the singular terms are correlated with UV divergent structure, which shows up the essential aspects of the quantum mechanical dynamics of NCYM theory not shared by NC-QED theory. The similar analysis for NCYM on the finite volume torus is interesting to further clearly the point, as was done for the analysis of UV divergence [4, 5]. It is also important to know the special role of supersymmetry which can control UV structure in the noncommutative system which is naturally obtained in the context of superstring theory [17]. The above connection of UV and infrared limit invokes us about the string theory together with the connection of gauge theory to the interacting string theory [13, 18, 19]. The problem is that the perturbative string theory is absent of UV divergence as insured by its moduli invariance while the noncommutative theory has UV divergence generally. Then supersymmetry will play the key role. Such a subject should be further investigated similarly to Ref. [10] attempting to construct the string theory which does not refer to any world-sheet points of view.

Acknowledgements

The author thanks greatly S. Iso for discussion and suggestion at frequent times and reading manuscript. He also thanks L. Susskind for pointing out the serious error in the result at the stage of Ref. [12], and N. Ishibashi, Y. Kitazawa, K. Okuyama and F. Sugino for learning about noncommutative theory, and the colleagues at KEK for sharing common interests in this and other various topics at a weekly informal meeting.

Appendix

Here to illustrate the convergence of the various nonplanar portions of diagrams, the ghost loop contribution in Fig. 3(a) to the vacuum polarization of the gauge field is explicitly evaluated. The application of Feynman rule in Fig. 1 gives to the associated contribution under the

dimensional regularization

$$i\Pi_{\text{gh}}^{\mu\nu}(q) = -i(2g)^2 \int \frac{d^d k}{i(2\pi)^d} \sin^2\left(\frac{1}{2}k \cdot C \cdot q\right) \frac{(k - q/2)^\mu (k + q/2)^\nu}{((k - q/2)^2 + i\epsilon)((k + q/2)^2 + i\epsilon)}. \quad (40)$$

Schwinger parameterization of the propagator [16]

$$\frac{1}{k^2 - m^2 + i\epsilon} = - \int_0^\infty i d\alpha e^{i\alpha(k^2 - m^2 + i\epsilon)}, \quad (41)$$

will enable us to carry out the integration over loop momentum k including the phase factor. Eq. (40) gives

$$i\Pi_{\text{gh}}^{\mu\nu}(q) = -i \frac{(2g)^2}{2} \int_0^\infty i d\alpha_+ \int_0^\infty i d\alpha_- \left[I^{\mu\nu}(0, q) - \frac{1}{2} (I^{\mu\nu}(1, q) + I^{\mu\nu}(-1, q)) \right], \quad (42)$$

where

$$I^{\mu\nu}(\eta, q) = p_-^\mu p_+^\nu \int \frac{d^d k}{i(2\pi)^d} \exp \left[i \left\{ \alpha_+ \left(k + \frac{q}{2} \right)^2 + \alpha_- \left(k - \frac{q}{2} \right)^2 + \eta k \cdot \tilde{q} + z_+ \cdot \left(k + \frac{q}{2} \right) + z_- \cdot \left(k - \frac{q}{2} \right) \right\} \right] \Big|_{z_\pm \rightarrow 0}, \quad (43)$$

with $p_\pm^\mu = \frac{1}{i} \frac{\partial}{\partial z_{\pm\mu}}$ is a derivative operator and $\tilde{q}^\mu = C^{\mu\nu} q_\nu$. Then the integral can be brought into the form of Gaussian-type. Use of

$$\int \frac{d^d k}{i(2\pi)^d} e^{i\beta k^2} = \frac{1}{(4\pi\beta i)^{d/2}}, \quad (44)$$

and performing the derivative operation yields

$$I^{\mu\nu}(\eta, q) = \frac{1}{(4\pi\beta i)^{d/2}} \exp \left[-i \left\{ \frac{\alpha_+ + \alpha_-}{\beta} (-q^2) + \frac{1}{\beta} \frac{\eta^2 \tilde{q}^2}{4} \right\} \right] \times \left[\frac{i}{2\beta} g^{\mu\nu} - \frac{\alpha_+ + \alpha_-}{\beta^2} q^\mu q^\nu + \frac{\eta^2}{4} \tilde{q}^\mu \tilde{q}^\nu + (\text{terms linear in } \eta) \right], \quad (45)$$

where β is $\beta = \alpha_+ + \alpha_-$. Thus the expression for the ghost loop contribution becomes

$$i\Pi_{\text{gh}}^{\mu\nu}(q) = -i \frac{(2g)^2}{2} \int_0^\infty i d\alpha_+ \int_0^\infty i d\alpha_- \frac{1}{(4\pi\beta i)^{d/2}} \exp \left(-i \frac{\alpha_+ + \alpha_-}{\beta} (-q^2) \right) \times \left\{ \left(1 - e^{-\frac{i}{\beta} \frac{\tilde{q}^2}{4}} \right) \left(\frac{i}{2\beta} g^{\mu\nu} - \frac{\alpha_+ + \alpha_-}{\beta^2} q^\mu q^\nu \right) - e^{-\frac{i}{\beta} \frac{\tilde{q}^2}{4}} \frac{1}{4\beta^2} \tilde{q}^\mu \tilde{q}^\nu \right\}. \quad (46)$$

The term not proportional to $e^{-\frac{i}{\beta}\frac{\tilde{q}^2}{4}}$ in the bracket of eq. (46) gives a planar contribution and is UV divergent.

The nonplanar contributions in (46), on the other hand, are convergent. Indeed the first term with nontrivial phase in eq. (46) becomes for $d = 4$ (\tilde{q} is space-like four momentum.)

$$\begin{aligned} & \int_0^\infty i d\alpha_+ \int_0^\infty i d\alpha_- \frac{1}{(4\pi\beta i)^{d/2}} \frac{1}{\beta} \exp \left[-i \left\{ \frac{\alpha_+ \alpha_-}{\beta} (-q^2) + \beta \mu^2 + \frac{1}{\beta} \frac{\tilde{q}^2}{4} \right\} \right] \\ &= i \frac{1}{16\pi^2} \int_0^1 d\alpha_+ \left\{ \alpha_+ (1 - \alpha_+) (-q^2) + \mu^2 \right\} \\ & \quad \times \int_0^\infty \frac{d\rho}{\rho^2} \exp \left[-\rho - \frac{1}{\rho} \left(\frac{-\tilde{q}^2}{4} \right) \left\{ \alpha_+ (1 - \alpha_+) (-q^2) + \mu^2 \right\} \right], \end{aligned} \quad (47)$$

where the small mass μ for the FP-ghost is introduced to regularize the infrared divergence, which will be found unnecessary hereafter. The UV behavior is characterized by the contribution from the integral around $\rho \sim 0$. Since the integral around the lower end in terms of $\lambda = 1/\rho$ becomes

$$\int_0^a \frac{d\rho}{\rho^n} \exp \left(-\rho - \frac{1}{\rho} a^2 \right) = \int_{1/a}^\infty d\lambda \lambda^{n-2} \exp \left(-a^2 \lambda - \frac{1}{\lambda} \right), \quad (48)$$

nonzero \tilde{q}^2 avoids the integral to diverge around $\rho \sim 0$ ($\lambda \rightarrow \infty$) for any power-like singularity. Actually the modified Bessel function of the second order writes the integral for $a > 0$

$$\int_0^\infty \frac{d\rho}{\rho^{n+1}} \exp \left(-\rho - \frac{1}{\rho} a^2 \right) = \left(-\frac{1}{2a} \frac{d}{da} \right)^n [2K_0(2a)]. \quad (49)$$

The asymptotic expansion of $K_0(x)$ around $x \sim 0$ is known [15] as

$$K_0(2a) = \sum_{k=0}^\infty \frac{a^{2k}}{(k!)^2} (-\ln(a) + \psi(k+1)), \quad (50)$$

where $\psi(z) = d\ln\Gamma(z)/dz$ becomes for an integer $z = n$

$$\psi(1) = -\gamma_E, \quad \psi(n) = -\gamma_E + \sum_{k=1}^{n-1} \frac{1}{k} \quad (n \geq 2), \quad (51)$$

with Euler constant γ_E . Eqs. (49) and (50) give the asymptotic expansion for the integrals entering in our one-loop calculus

$$\begin{aligned} \int_0^\infty \frac{d\rho}{\rho} \exp \left(-\rho - \frac{1}{\rho} a^2 \right) &= -\ln(a^2) \left(1 + a^2 + \mathcal{O}(a^4) \right) - 2\gamma_E + (-2\gamma_E + 2)a^2 \\ &\quad + \mathcal{O}(a^4), \\ \int_0^\infty \frac{d\rho}{\rho^2} \exp \left(-\rho - \frac{1}{\rho} a^2 \right) &= \ln(a^2) \left(1 + \frac{1}{2} a^2 + \mathcal{O}(a^4) \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{a^2} + (2\gamma_E - 1) + \left(\gamma_E - \frac{5}{4}\right) a^2 + \mathcal{O}(a^4), \\
\int_0^\infty \frac{d\rho}{\rho^3} \exp\left(-\rho - \frac{1}{\rho} a^2\right) &= -\ln(a^2) \left(\frac{1}{2} + \frac{1}{6} a^2 + \mathcal{O}(a^4)\right) + \frac{1}{a^4} - \frac{1}{a^2} \\
& + \left(-\gamma_E + \frac{3}{4}\right) + \left(-\frac{1}{3}\gamma_E + \frac{39}{36}\right) a^2 + \mathcal{O}(a^4). \quad (52)
\end{aligned}$$

where $n = 2$ to use for eq. (47).

References

- [1] A. Gonzalez-Arroyo and M. Okawa, “A twisted model for large N lattice gauge theory”, Phys. Lett. **B120**, 174 (1983); “The twisted Eguchi-Kawai model: a reduced model for large N lattice gauge theory,” Phys. Rev. **D27**, 2397 (1983).
- [2] T. Filk, “Divergencies in a field theory on quantum space”, Phys. Lett. **B376**, 53 (1996).
- [3] C. P. Martin and D. Sanchez-Ruiz, “The one-loop UV divergent structure of U(1) Yang-Mills theory on noncommutative R^4 ”, Phys. Rev. Lett. **83**, 476 (1999), hep-th/9903077.
- [4] M. M. Sheikh-Jabbari, “One loop renormalizability of supersymmetric Yang-Mills theories on noncommutative two-torus”, JHEP 9906 (1999) 015, hep-th/9903107.
- [5] T. Krajewski and R. Wulkenhaar, “Perturbative quantum gauge fields on the noncommutative torus”, hep-th/9903187.
- [6] D. Bigatti and L. Susskind, ”Magnetic fields, branes and noncommutative geometry”, hep-th/9908056.
- [7] J. M. Maldacena and J. G. Russo, “Large N limit of non-commutative gauge theories,” JHEP **09**, 025 (1999), hep-th/9908134.
- [8] M. Alishahiha, Y. Oz and M.M. Sheikh-Jabbari, “Supergravity and large N noncommutative field theories,” JHEP **11**, 007 (1999), hep-th/9909215.
- [9] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, “Wilson loops in noncommutative Yang-Mills”, hep-th/9910004.
- [10] S. Minwalla, M. V. Raamsdonk and N. Seiberg, “Noncommutative Perturbative Dynamics”, hep-th/9912072.
- [11] I. Ya. Aref’eva, D. M. Belov and A. S. Koshelev, “Two-loop diagrams in noncommutative ϕ_4^4 theory”, hep-th/9912075.
- [12] M. Hayakawa, “Perturbative analysis on infrared aspects of noncommutative QED on R^4 ”, hep-th/9912094.

- [13] G. 't Hooft, “A planar diagram theory for strong interactions”, Nucl. Phys. **B72**, 461 (1974).
- [14] T. Eguchi and H. Kawai, “Reduction of dynamical degrees of freedom in the large N gauge theory”, Phys. Rev. Lett. **48**, 1063 (1982).
- [15] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, 1972).
- [16] C. Itzykson and J-B. Zuber, *Quantum Field Theory* (McGraw-Hill, 1985).
- [17] M. Li, “Strings from IIB matrices”, Nucl.Phys. B499 (1997) 149-158, hep-th/9612222; A. Connes, M. R. Douglas and A. Schwarz, “Noncommutative geometry and matrix theory: Compactification on tori”, JHEP **02**, 003 (1998) hep-th/9711162; M. R. Douglas and C. Hull, “D-branes and the noncommutative torus”, JHEP **02**, 008 (1998), hep-th/9711165; H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, “Noncommutative Yang-Mills in IIB matrix model”, hep-th/9908141; N. Seiberg and E. Witten, “String theory and noncommutative geometry”, JHEP **09**, 032 (1999), hep-th/9908142; K. Okuyama, “A path integral representation of the map between commutative and noncommutative gauge fields”, hep-th/9910138; J. Ambjorn, Y. M. Makeenko, J. Nishimura and R. J. Szabo, “Finite N matrix models of noncommutative gauge theory”. JHEP **9911**, 029 (1999), hep-th/9911041.
- [18] A. M. Polyakov, “Gauge Fields and Strings” (Harwood Academic Publishers, 1987).
- [19] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: a conjecture”, Phys. Rev. **D55**, 5112 (1997), hep-th/9610043; V. Periwal, “Matrices on a point as the theory of everything”, Phys. Rev. **D55**, 1711 (1997), hep-th/9611103; N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A large-N reduced model as superstring”, Nucl. Phys. **B498**, 467 (1997), hep-th/9612115; L. Motl, “Proposals on nonperturbative superstring interactions”, hep-th/9701025; R. Dijkgraaf, E. Verlinde and H. Verlinde, “Matrix string theory”, Nucl. Phys. **B500**, 43 (1997), hep-th/9703030.